Lec - 2 Differential Geometry SIMAR NARULA (1)

Def: Two charts (U, x) & (V, y) are $\underline{C^\infty \text{ compatible}}$ if

(a) $U \cap V = \phi$

(b) $U \cap V \neq \phi$ & $x \cdot y^{-1} : \underset{\cap \atop \mathbb{R}^n}{y(U \cap V)} \to \underset{\cap \atop \mathbb{R}^n}{x(U \cap V)}$

& $y \cdot x^{-1} : \underset{\cap \atop \mathbb{R}^n}{x(U \cap V)} \to \underset{\cap \atop \mathbb{R}^n}{y(U \cap V)}$

are smooth (C^∞)

→ An Atlas $\mathcal{A}$ is C^∞ (smooth) if all its charts are C^∞ compatible

Def^n: A $\underline{\text{smooth manifold}}$ $(M, O, \mathcal{A})$ is a triple where (M, O) is a topological manifold & $\mathcal{A}$ is a smooth atlas.

Smooth Maps →

Let $\phi : M \to N$ be a map where $(M, O_M, \mathcal{A}_M)$ & $(N, O_N, \mathcal{A}_N)$ are smooth manifolds

$\underline{\phi \text{ is said to be smooth at } p \in M}$ if for some chart $(U, x) \in \mathcal{A}_M$ with $p \in U$ & $(V, y) \in \mathcal{A}_N$ with $\phi(p) \in V$ the map $y \cdot \phi \cdot x^{-1}$ is C^∞ (smooth) at $x(p) \in x(U) \subseteq \mathbb{R}^{\dim M}$ in the usual sense

$$
\begin{array}{ccc}
U \subseteq M & \xrightarrow{\phi} & V \subseteq N \\
\downarrow x & & \downarrow y \\
x(U) \subseteq \mathbb{R}^{\dim M} & \xrightarrow{y \circ \phi \circ x^{-1}} & y(V) \subseteq \mathbb{R}^{\dim N}
\end{array}
$$

Remark: The definition of differentiability is well-defined (i.e. it is chart independent)

Proof: We want to show that if $y \circ \phi \circ x^{-1}$ is differentiable at $x(p)$ for some $(U, x) \in \mathcal{A}_M$ with $p \in U$ & $(V, y) \in \mathcal{A}_N$ with $\phi(p) \in V$ then $\tilde{y} \circ \phi \circ \tilde{x}^{-1}$ is differentiable at $\tilde{x}(p)$ for all charts $(\tilde{U}, \tilde{x}) \in \mathcal{A}_M$ with $p \in \tilde{U}$ & $(\tilde{V}, \tilde{y}) \in \mathcal{A}_N$ with $\phi(p) \in \tilde{V}$

$$
\begin{tikzcd}
\tilde{x}(U \cap \tilde{U}) \subseteq \mathbb{R}^{\dim M} \arrow[r, "\tilde{y} \circ \phi \circ \tilde{x}^{-1}"] & \tilde{y}(V \cap \tilde{V}) \subseteq \mathbb{R}^{\dim N} \\
U \cap \tilde{U} \subseteq M \arrow[u, "\tilde{x}"] \arrow[d, "x"] \arrow[r, "\phi"] & V \cap \tilde{V} \subseteq N \arrow[u, "\tilde{y}"] \arrow[d, "y"] \\
x(U \cap \tilde{U}) \subseteq \mathbb{R}^{\dim M} \arrow[r, "y \circ \phi \circ x^{-1}"] \arrow[uu, bend left=60, "\tilde{x} \circ x^{-1}"] & y(V \cap \tilde{V}) \subseteq \mathbb{R}^{\dim N} \arrow[uu, bend right=60, "\tilde{y} \circ y^{-1}"']
\end{tikzcd}
$$

As ~~x &~~ (U, x) & $(\tilde{U}, \tilde{x}) \in \mathcal{A}_M$ which is C^∞ so $\tilde{x} \circ x^{-1}$ is smooth. Similarly $\tilde{y} \circ y^{-1}$ is smooth.

Now notice we can write

$$\tilde{y} \circ \phi \circ \tilde{x}^{-1} = (\tilde{y} \circ y^{-1}) \circ (y \circ \phi \circ x^{-1}) \circ (\tilde{x} \circ x^{-1})^{-1}$$

and since composition of smooth maps is smooth, we are done!

→ If the map ϕ is smooth for all points over its domain then it is called a <u>smooth map</u>

Diffeomorphism → $\phi : M \to N$ is a diffeo if it is a bijection b/w smooth manifolds (M, O_M, A_M) and (N, O_N, A_N) & both ϕ & ϕ^{-1} are smooth.

→ Smooth Manifolds (M, O_M, A_M) & (N, O_N, A_N) are said to be diffeomorphic if there is a ~~dm~~ diffeomorphism b/w them.

→ Diffeomorphisms are structure preserving maps b/w smooth manifolds

Tensors

Say we have a finite dimensional vector space $(V, +, \cdot)$ over field F with its basis e_i

We know that the dual space is defined as followed

$$V^* = \{ \omega : V \to F \mid \omega \text{ is linear} \}$$

~~&~~ $\omega \in V^*$ are called covectors

Note above we have just defined the set V^*. To make it a vector space we also need to define addition & scalar multiplication over it

$(V^*, +_{V^*}, \cdot_{V^*})$ is a vector space with $+_{V^*} : V^* \times V^* \to V^*$

$\cdot_{V^*} : F \times V^* \to V^*$

define pointwise as follows →

$$(\omega_1 +_{V^*} \omega_2)(v) := \omega_1(v) +_F \omega_2(v)$$

$$(\lambda \cdot_{V^*} \omega)(v) := \cancel{\lambda t} \; \lambda \cdot_F \omega(v)$$

We can also verify that all the axioms of vector field are obeyed. This is left as an exercise to the reader.

Eg

Now consider the following set

$$S = V \times V$$

So $(v_1, v_2) \in S$ if $v_1, v_2 \in V$

Can S be made into a vector space? $(S, +_S, \cdot_S)$

Yes with foll^n pointwise def^n of $+_S$ & $\cdot_S$

$$(v_1, v_2) +_S (v_3, v_4) := (v_1 +_V v_3, v_2 +_V v_4)$$

$$\lambda \cdot_S (v_1, v_2) := (\lambda \cdot_V v_1, \lambda \cdot_V v_2)$$

Now let us talk about bilinear functions on S which map to F

Let $f : V \times V \longrightarrow F$ such that f is bilinear

Bilinear means linear in each index

$$f(\lambda_1 v_1 + \lambda_2 v_2, v_3) = \lambda_1 f(v_1, v_3) + \lambda_2 f(v_2, v_3)$$

$$f(v_1, \lambda_2 v_2 + \lambda_3 v_3) = \lambda_2 f(v_1, v_2) + \lambda_3 f(v_1, v_3)$$

Now consider the set of all such functions f

$$X = \{ f : S \to F \mid f \text{ is bilinear} \}$$

Can we make X into a vector space?

Yes [Just like we made dual space into a vector space]

$$(f +_X g)(v_1, v_2) := f(v_1, v_2) +_F g(v_1, v_2)$$

$$(\lambda \cdot_X f)(v_1, v_2) := \lambda \cdot_F f(v_1, v_2)$$

So yes $(X, +_X, \cdot_X)$ is a vector space

Now let us consider the set

$$\underbrace{V^* \times V^* \times \dots \times V^*}_{p} \times \underbrace{V \times V \times \dots \times V}_{q}$$

~~As we see~~ This set can be made into a vector space [Similar to how we made $V \times V$ a vector space]

Now let us consider maps from this set to F

Def A <u>Tensor</u> of type (p,q) is a multilinear map $T : \underbrace{V^* \times V^* \times \dots \times V^*}_{p \text{ copies}} \times \underbrace{V \times V \times \dots V}_{q \text{ copies}} \to F$

The set of all (p,q) type tensor is called $\underline{T^p_q V}$

$T^p_q V$ is also a vector space [As X was made a vector space in the last Example]

Eg A $(0,1)$ tensor is a covector & $T^0_1 V = V^*$

A $(1,0)$ tensor is a vector & $T^1_0 V = V$

[Note that dimension of V is finite]

Eg $T \in T^1_1 V$; $\omega, \eta \in V^*$ & $v, w \in V$

$$T(a\omega + b\eta, cv + dw) = ac\,T(\omega, v) + ad\,T(\omega, w) + bc\,T(\eta, v) + bd\,T(\eta, w)$$

Def. Let $T \in T^p_q V$ & $S \in T^r_s V$. Then we define the tensor product $T \otimes S$ as follows –

1. $T \otimes S \in T^{p+r}_{q+s} V$
2. ~~For any $\omega_i \in V^*$ & any $v_i \in V$. The tensor product is defined pointwise~~

$$(T \otimes S)(\omega_1, \omega_2, \dots \omega_p, \omega_{p+1}, \dots \omega_{p+r}, v_1, \dots v_q, \dots v_{q+s}) := T(\omega_1, \omega_2, \dots \omega_p, v_1, \dots v_q) \cdot S(\omega_{p+1}, \dots \omega_{p+r}, v_{q+1}, \dots$$

2. The tensor product is defined pointwise as→

$$(T\otimes S)(\omega_1,\omega_2,\dots\omega_p,\omega_{p+1},\dots\omega_{p+r},v_1,v_2,\dots v_q,v_{q+1},\dots v_{q+s}) :=$$

$$T(\omega_1,\omega_2,\dots \omega_p,v_1,\dots v_q)\cdot S(\omega_{p+1},\omega_{p+2},\dots\omega_{p+r},v_{q+1},\dots v_{q+s})$$

multiplication in F

where $\omega_i \in V^*$ & $v_i \in V$ & $\omega_i \neq \omega_j$ for ~~any~~ $i \neq j$
$v_i \neq v_j$

Note the ω_i & v_i are covectors & vectors themselves. They are not the components of a covector/vector

Note In general $T\otimes S \neq S \otimes T$

Claim: Given a basis e_i of V & the dual basis f^j of V^* ($f^j(e_i)=\delta^j{}_i$) we can construct a basis of $T^p{}_q V$ by using tensors which consist of all tensors of the form→

$$e_{\mu_1}\otimes e_{\mu_2}\otimes \dots \otimes e_{\mu_p}\otimes f^{\nu_1}\otimes f^{\nu_2}\otimes\dots\otimes f^{\nu_q}$$

If dim of V is n then there will be n^{p+q} basis tensor in all [i.e. $\dim T^p_q V = (\dim V)^{p+q}$]

Proof ① ~~Sp~~ The set is spanning:

Consider some arbitrary Tensor $T \in T^p_q V$

Construct a tensor S s.t.

$$S = S^{\mu_1\mu_2\dots\mu_p}{}_{\nu_1\nu_2\dots\nu_q}\, e_{\mu_1}\otimes\dots e_{\mu_p}\otimes f^{\nu_1}\otimes\dots f^{\nu_q}$$

where

$$S^{\mu_1\mu_2\dots\mu_p}{}_{\nu_1\nu_2\dots\nu_q} := T(f^{\mu_1},\dots,f^{\mu_p},e_{\mu_1},\dots e_{\mu_q})$$

Now we will prove that ~~these~~ tensor T&S are the same.

To show this consider action of S on arbitrary vectors & covectors ~~$\alpha^{(i)} \in V^*$~~ &, $\omega^{(i)} \in V^*$ (i goes from 1 to p) & $v_{(i)} \in V$ (i goes from 1 to q)

~~$S(\alpha'$~~ Note that we can write the vectors & covectors in terms of their components,

$$v_{(i)} = v_{(i)}^{j}\, e_j$$

$$\& \quad \omega^{(i)} = \omega^{(i)}_{j}\, f^{j}$$

Now,
$$S(\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(p)}, v_{(1)}, v_{(2)}, \ldots, v_{(q)})$$

$$= \cancel{T(} S^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q}\,(e_{\mu_1}\otimes\ldots\otimes e_{\mu_p}\otimes f^{\nu_1}\otimes\ldots\otimes f^{\nu_q})(\omega^{(1)}, \omega^{(2)}, \ldots \omega^{(p)}, v_{(1)}, \ldots v_{(q)})$$

$$= S^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q}\,[e_{\mu_1}(\omega^{(1)})][e_{\mu_2}(\omega^{(2)})]\ldots[e_{\mu_p}(\omega^{(p)})]\,[f^{\nu_1}(v_{(1)})][f^{\nu_2}(v_{(2)})]\ldots[f^{\nu_q}(v_{(q)})]$$

Now note

$$e_{\mu_i}(\omega^{(i)}) = e_{\mu_i}(\omega^{(i)}_j f^j) = \omega^{(i)}_j\, e_{\mu_i}(f^j) = \omega^{(i)}_j\,\delta^j_{\mu_i} = \omega^{(i)}_{\mu_i}$$

Similarly $f^{\nu_q}(v_{(q)}) = v^{\nu_q}_{(q)}$

$$= S^{\mu_1\mu_2\ldots\mu_p}{}_{\nu_1\nu_2\ldots\nu_q}\cdot \underbrace{\omega^{(1)}_{\mu_1}\cdot\omega^{(2)}_{\mu_2}\cdots\omega^{(p)}_{\mu_p}\cdot v^{\nu_1}_{(1)}\cdots v^{\nu_q}_{(q)}}$$

~~$= T(f^{\mu_1}, f^{\mu_2},$~~

$$= T(f^{\mu_1}, f^{\mu_2}, \ldots f^{\mu_p}, e_{\nu_1}, e_{\nu_2}, \ldots e_{\nu_q})\cdot(\qquad)$$

$$= \cancel{T(f}$$

$$= T(\omega^{(1)}_{\mu_1} f^{\mu_1}, \omega^{(2)}_{\mu_2} f^{\mu_2}, \dots \omega^{(p)}_{\mu_p} f^{\mu_p}, v^{\nu_1}_{(1)} e_{\nu_1}, \dots v^{\nu_q}_{(q)} e_{\nu_q})$$

[Using multilinearity of T]

$$= T(\omega^{(1)}, \omega^{(2)}, \dots \omega^{(p)}, v_{(1)}, v_{(2)}, \dots v_{(q)})$$

$\therefore$ for any arbitrary choice of $\omega^{(1)}, \omega^{(2)}, \dots \omega^{(p)}, v_{(1)}, \dots v_{(q)}$

we have $S(\omega^{(1)}, \omega^{(2)}, \dots v_{(q)}) = T(\omega^{(1)}, \omega^{(2)}, \dots v_{(q)})$

$\therefore$ we have $S = T$

2. Now we have to show linear independence.

Suppose $C^{\mu_1 \mu_2 \dots \mu_p}_{\nu_1 \nu_2 \dots \nu_q} e_{\mu_1} \otimes \dots \otimes f^{\nu_q} = 0$

Act this tensor on $(f^{\alpha_1}, f^{\alpha_2}, \dots f^{\alpha_p}, e_{\beta_1}, \dots e_{\beta_q})$

we get

$$C^{\mu_1 \mu_2 \dots \mu_p}_{\nu_1 \nu_2 \dots \nu_q} [e_{\mu_1}(f^{\alpha_1})][e_{\mu_2}(f^{\alpha_2})] \dots \dots [f^{\nu_q}(e_{\beta_q})] = 0$$

$$\Rightarrow C^{\mu_1 \mu_2 \dots \mu_p}_{\nu_1 \nu_2 \dots \nu_q} \delta^{\alpha_1}_{\mu_1} \delta^{\alpha_2}_{\mu_2} \dots \delta^{\alpha_p}_{\mu_p} \delta^{\nu_1}_{\beta_1} \dots \delta^{\nu_q}_{\beta_q} = 0$$

$$\Rightarrow C^{\alpha_1 \alpha_2 \dots \alpha_p}_{\beta_1 \beta_2 \dots \beta_q} = 0$$

Hence all coeff are zero.

$\therefore$ The tensors are linearly independent

Hence they form a basis

Notation: We will denote the tensor space $T^p_q V$ by

$$T^p_q V = \underbrace{V \otimes V \otimes \dots \otimes V}_{p} \otimes \underbrace{V^* \otimes V^* \otimes \dots \otimes V^*}_{q}$$

[This ~~is~~ notation is motivated by the basis of $T^p_q V$]

So any tensor $T \in T^p_q V$ can be written as

$$T = T^{i_1 i_2 \dots i_p}_{\quad j_1 j_2 \dots j_q} e_{i_1} \otimes e_{i_2} \otimes \dots e_{i_p} \otimes f^{j_1} \otimes f^{j_2} \otimes \dots f^{j_q}$$

Here $T^{i_1 i_2 \dots i_p}_{\quad j_1 j_2 \dots j_q}$ are the components of the tensor.

We also have

$$T^{i_1 i_2 \dots i_p}_{\quad j_1 j_2 \dots j_q} = \cancel{T(f_{i_1}, f_{i_2}, \dots f_{i_p}, \dots e_j)}$$

$$= T(f^{i_1}, f^{i_2}, \dots f^{i_p}, e_{j_1}, e_{j_2}, \dots e_{j_q})$$

Eg $T \in T^1_1 V = V \otimes V^*$

$$T(\omega, \upsilon) = T(\omega_i f^i, \upsilon^j e_j)$$

$$= \omega_i \upsilon^j \, T(f^i, e_j)$$

$$= \omega_i \upsilon^j \, T^i_{\ j}$$

~~Alter~~ $T(\omega, \upsilon) = T^i_{\ j} \, e_i \otimes f^j (\omega, \upsilon)$

$$= T^i_{\ j} \, e_i(\omega) \cdot f^j(\upsilon)$$

$$= T^i_{\ j} \, e_i(\omega_a f^a) \, f^j(\upsilon^b e_b)$$

$$= T^i_{\ j} \, \omega_a \upsilon^b \, e_i(f^a) \, f^j(e_b)$$

$$= T^i_{\ j} \, \omega_a \upsilon^b \, \delta^a_{\ i} \delta^j_{\ b}$$

$$= T^i_{\ j} \, \omega_i \upsilon^j$$

Change of basis →

Suppose we change the basis of V from $\{e_i\}$ to $\{\tilde{e}_i\}$

we have $\tilde{e}_i = A^j{}_i \, e_j$

& $e_i = B^j{}_i \, \tilde{e}_j$

A & B are similarity matrices. From linear algebra we know that $B = A^{-1}$. (we won't prove it here)

Say $v \in V$

$$v = v^i e_i = \tilde{v}^i \tilde{e}_i$$

$$\Rightarrow \tilde{v}^i \tilde{e}_i = v^i e_i$$
$$= v^i B^j{}_i \, \tilde{e}_j$$
$$= (B^j{}_i v^i) \, \tilde{e}_j \Rightarrow \tilde{v}^j = B^j{}_i v^i$$

Similarly $v^i = A^j{}_i \tilde{v}^i$

~~So the components of vectors transform ~~

∴ The transformation of vector components is inverse of the transformation of basis.

Now $f^i(e_j) = \delta^i{}_j = \tilde{f}^i(\tilde{e}_j)$

From this we see $\tilde{f}^i = (A^{-1})^i{}_j \, f^j$

$$= B^i{}_j \, f^j$$

Similarly $\omega = \tilde{\omega}_i \tilde{f}^i = \omega_i f^i$ gives

$$\tilde{\omega}_i = A^j{}_i \, \omega_j$$

Transformation of Tensor components →

$$T = T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}\, e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_p} \otimes f^{j_1} \otimes f^{j_2} \otimes \dots \otimes f^{j_q}$$

$$= T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}\, B^{i_1'}_{i_1} \tilde{e}_{i_1'} \otimes B^{i_2'}_{i_2} \tilde{e}_{i_2'} \otimes \dots \otimes B^{i_p'}_{i_p} \tilde{e}_{i_p'} \otimes A^{j_1}_{j_1'} \tilde{f}^{j_1'} \otimes A^{j_2}_{j_2'} \tilde{f}^{j_2'} \otimes \dots \otimes A^{j_q}_{j_q'} \tilde{f}^{j_q'}$$

$$= T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}\, B^{i_1'}_{i_1} B^{i_2'}_{i_2} \dots B^{i_p'}_{i_p} A^{j_1}_{j_1'} \dots A^{j_q}_{j_q'}\; \tilde{e}_{i_1'} \otimes \tilde{e}_{i_2'} \otimes \dots \tilde{e}_{i_p'} \otimes \tilde{f}^{j_1'} \otimes \dots \tilde{f}^{j_q'}$$

$$\therefore\ \tilde{T}^{i_1' i_2' \dots i_p'}_{j_1' j_2' \dots j_q'} = T^{i_1 i_2 \dots i_p}_{j_1 j_2 \dots j_q}\, B^{i_1'}_{i_1} B^{i_2'}_{i_2} \dots B^{i_p'}_{i_p} A^{j_1}_{j_1'} \dots A^{j_q}_{j_q'}$$

Note:

1. Remember the basis vectors & the components of covectors transform in same fashion and the components of vectors & basis of covectors transform in the inverse fashion to former
2. The Transformation rules are important because vectors, covectors & tensors are defined in physics via transformation rules
3. From now on, to check if an object is a vector/covector/tensor we will ~~also~~ see how its components ~~it~~ transforms under basis change. If they match with transformation rules of an object then it is indeed that object!

TANGENT SPACES

~~Let~~ We will now come back to discussion of smooth manifolds M

Def^n The set of all smooth functions $f: M \to \mathbb{R}$ is called $\underline{C^\infty(M)}$

$$C^\infty(M) = \{ f : M \to \mathbb{R} \mid f \text{ is smooth} \}$$

YOU CAN SKIP THIS PART →

We can define ~~+~~ $+$ & $\cdot$ operations on $C^\infty(M)$

[~~It~~ Note that this space does not form a vector space (because there is no commutativity in $C^\infty(M)$)]

Def^n A smooth curve on M is a $\underline{\text{smooth map}}$ $\gamma : \mathbb{R} \to M$ where $\mathbb{R}$ is understood as a 1 Dimensional manifold

Def Let $\gamma : \mathbb{R} \to M$ be a smooth curve through $p \in M$; w.l.o.g let $\gamma(0) = p$

The $\underline{\text{directional derivative operator}}$ at p along γ is the linear map

$$X_{\gamma,p} : C^\infty(M) \xrightarrow{\sim} \mathbb{R} \qquad [\sim \text{ is for linear}]$$

$$X_{\gamma,p}(f) := (f \circ \gamma)'(0)$$

↑ where $0 = \gamma(p)$

$$\mathbb{R} \xrightarrow{\gamma} M \xrightarrow{f} \mathbb{R}$$
$$t \qquad \gamma(t) \qquad f(\gamma(t))$$

$$f \circ \gamma : \mathbb{R} \to \mathbb{R}$$ (so we can take derivative in the usual sense)

$$X_{\gamma,p}(f) := (f \circ \gamma)'(0)$$

Directional derivative

So the directional derivative acts on functions to give a real number.

But its existence is not dependent on the function its acts on.

Remark - $X_{\gamma,p}$ is called the tangent vector to the curve γ at point $p \in M$. Intutively, $X_{\gamma,p}$ is the velocity of γ at p. Consider the curve $\delta(t) := \gamma(2t)$ which is the same curve parametrized twice as fast. For any $f \in C^\infty(M)$, we have

$$X_{\delta,p}(f) = (f \circ \delta)'(0) = 2(f \circ \gamma)'(0) = 2X_{\gamma,p}(f)$$

(By using chain rule)

$\therefore$ $X_{\gamma,p}$ Scales as velocity should

<u>Def</u> $T_pM = \{ X_{\gamma,p} \mid \gamma \text{ is a smooth curve through } p \}$

i.e. T_pM is the set of all ~~das~~ tangent vectors at point $p \in M$. T_pM is called the tangent space at point p.

Claim: T_pM can be made into a vector space

$(T_pM, \oplus, \odot)$

we define $\oplus$ & $\odot$ pointwise as follows →

$$(X_{\gamma,p} \oplus X_{\delta,p})(f) := X_{\gamma,p}(f) +_{\mathbb{R}} X_{\delta,p}(f)$$

$$(\lambda \odot X_{\gamma,p})(f) := \lambda \cdot_{\mathbb{R}} X_{\gamma,p}(f)$$

But the above defn do not imply closure

So we need to show ~~∃ γ, δ smooth cu~~ that ∃ μ, σ smooth curves s.t.

① $X_{\gamma,p}(f) + X_{\delta,p}(f) = X_{\sigma,p}(f)$

② $\lambda \cdot X_{\gamma,p} = X_{\mu,p}(f)$

After we have showed this then only we can say that $(T_pM, \oplus, \odot)$ is a vector space

Proof of ① → Let (U,x) be a chart on M with $p \in U$

Define $\sigma(t) := x^{-1}((x \circ \gamma)(t) + (x \circ \delta)(t) - x(p))$

Note that $\underline{\sigma \text{ is smooth}}$ as it is constructed by addn & compn of smooth maps

$$\sigma(0) = x^{-1}((x \circ \gamma)(0) + (x \circ \delta)(0) - x(p))$$
$$= x^{-1}(x(p) + x(p) - x(p))$$
$$= x^{-1}(x(p))$$
$$= p$$

∴ σ passes through p

$$X_{\sigma,p}(f) := (f\circ\sigma)'(0)$$

$$= \left[f\circ x^{-1}\circ((x\circ\gamma)+(x\circ\delta)-x(p))\right]'(0)$$

$$= \left[\partial_a(f\circ x^{-1})(x(p))\right]\left(\left[(x^a\circ\gamma)+(x^a\circ\delta)-x^a(p)\right)'(0)$$

~~where~~ where we used the multivariable chain rule as $f\circ x^{-1} : \mathbb{R}^{\dim M} \to \mathbb{R}$

and $(x\circ\gamma + x\circ\delta - x(p)) : \mathbb{R} \to \mathbb{R}^d$

$$= \left[\partial_a(f\circ x^{-1})(x(p))\right]\left[(x^a\circ\gamma)'(0) + (x^a\circ\delta)'(0)\right]$$

[As derivative is a linear operation for $\mathbb{R} \to \mathbb{R}^d$]

$$= (f\circ x^{-1}\circ x\circ\gamma)'(0) + (f\circ x^{-1}\circ x\circ\delta)'(0)$$

[By distributive rule]

$$= (f\circ\gamma)'(0) + (f\circ\delta)'(0)$$

$$= X_{\gamma,p}(f) + X_{\delta,p}(f)$$

$$\therefore X_{\sigma,p}(f) = X_{\gamma,p}(f) + X_{\delta,p}(f)$$

$$\therefore X_{\gamma,p} \oplus X_{\delta,p} = X_{\sigma,p} \in T_pM$$

H.P.

Outline of Proof of ② → Take $\mu(t) = \gamma(\lambda t)$ ~~and~~

The completion of this proof is left as an exercise to the reader

Remark → The tangent vectors $X_{\gamma,p}$ satisfy <u>Leibnizarity</u>.

$$X_{\gamma,p}(f\cdot g) = X_{\gamma,p}(f)\, g(p) + f(p)\, X_{\gamma,p}(g)$$

(Observe, This is similar to chain rule; This is where the name directional "derivative" comes from)

The proof of this remark is ~~also~~ left as an exercise to the reader. Hint: Again use a local coordinate system (U, x) near p.

<u>Thm</u>: $\dim T_pM = \dim M = n$ (Say)

{ Note that the dimension on LHS & on RHS have very different meanings. On LHS it is the dimension of vector space T_pM which is the number of ~~t~~ vectors in a basis set. On RHS it is the dimension of manifold M (which is the dim of $\mathbb{R}^k$ on which the homeomorphisms x of (U, x) map to $x: U \to \mathbb{R}^k$). }

To prove this theorem, first we will construct n tangent vectors on T_pM, then we will show that any vector in T_pM can be written as a linear combn of them. Then we will show ~~that~~ they are linearly independent.

Take a chart (U, x) on M

(U, x) chart

$x : M \to \mathbb{R}$

$\boxed{x(p) = 0}$

$x(t) = (x^1(t), x^2(t), \ldots, x^{\dim m}(t))$

Define

$\gamma_1 : \mathbb{R} \to U$ s.t. $\gamma_1(0) = p$

$\gamma_1(t) = x^{-1} \circ (t, 0, 0, \ldots, 0)$

$\gamma_2 : \mathbb{R} \to U$ s.t. $\gamma_2(0) = p$

$\gamma_2(t) = x^{-1} \circ (0, t, \ldots, 0)$

i.e. $x^1(\gamma_1(t)) = t$

$x^2(\gamma_1(t)) = 0$

$x^3(\gamma_1(t)) = 0$

x

$x^a(\gamma_b(t)) = \delta^a_b\, t$

$\boxed{(x^a \circ \gamma_b)(t) = \delta^a_b\, t}$

$$X_{\gamma_a, p}(f) = (f \circ \gamma_a)'(x(p))$$

$$= (f \circ id \circ \gamma_a)'(x(p))$$

$$= (f \circ x^{-1} \circ x \circ \gamma_a)'(x(p))$$

$$[\partial_b(f \circ x^{-1})(x(p))]\ (x^b \circ \gamma_a)'(x(p))$$

$$[\partial_b(f \circ x^{-1})](x(p))\ \delta^a_b \qquad (\delta^a_b\, t)'(0) = \delta^a_b$$

$$= [\partial_a(f \circ x^{-1})](x(p))$$

$f \circ x^{-1}$

$$\mathbb{R}^n \xrightarrow{x^{-1}} U \xrightarrow{f} \mathbb{R}$$

$f \circ x^{-1}$

$$\therefore X_{\gamma_a, p}(f) = [\partial_a(f \circ x^{-1})](x(p))$$

Renaming

$$\left(\frac{\partial}{\partial x^a}\right)_p (f) := (\partial_a(f \circ x^{-1}))(x(p))$$

$$X_{\gamma_a, p} = \left(\frac{\partial}{\partial x^a}\right)_p$$

$$\left(\frac{\partial}{\partial x^a}\right)_p \in T_p M$$

↳ These form a basis of $T_p M$ where $a = 1, 2, \ldots, \dim M$

Pf Let $X \in T_p M$

So by def $\exists\ \sigma : I \longrightarrow M$: $\sigma(0) = p$, $I \subset \mathbb{R}$

and $X = X_{\sigma, p}$

$X_{\gamma_a, p}$

$\boxed{\partial_a(f \circ x^{-1})(x(p)) := \frac{\partial f}{\partial x^a}}$

$$X(f) = X_{\sigma, p}(f) = (f \circ \sigma)'(0)$$

$$= (f \circ x^{-1} \circ x \circ \sigma)'(0)$$

$$= [\partial_a(f \circ x^{-1})(x(p))]\ (x^a \circ \sigma)'(0)$$

$$= \left(\frac{\partial}{\partial x^a}(f)\right) \quad (x^a \circ \sigma)'(0)$$

$$= \underbrace{(x^a \circ \sigma)'(0)} \; \frac{\partial}{\partial x^a}(f)$$

$$= X^a \frac{\partial}{\partial x^a}(f)$$

$$\therefore \boxed{X = X^a \frac{\partial}{\partial x^a}}$$

where if $X = X_{\sigma,p}$

$$X^a = (x^a \circ \sigma)'(0)$$

Show linear independence of $\left(\partial/\partial x^a\right)_p$

To show linear independence, suppose that

$$\lambda^a \left(\frac{\partial}{\partial x^a}\right)_p = 0,$$

$X_{\gamma,p}(f) = (f\circ\gamma)'(0)$

for some scalars λ^a. Note that this is an operator equation, and the zero on the right hand side is the zero operator $0 \in T_pM$.

Recall that, given the chart (U, x), the coordinate maps $x^b : U \to \mathbb{R}$ are smooth, i.e. $x^b \in \mathcal{C}^\infty(U)$. Thus, we can feed them into the left hand side to obtain

$$\begin{aligned} 0 &= \lambda^a \left(\frac{\partial}{\partial x^a}\right)_p (x^b) \\ &= \lambda^a \, \partial_a(x^b \circ x^{-1})(x(p)) \\ &= \lambda^a \, \partial_a(\mathrm{proj}_b)(x(p)) \\ &= \lambda^a \delta^b_a \\ &= \lambda^b \end{aligned}$$

$\left(\frac{\partial}{\partial x^a}\right)_p f = \partial_a(f\circ x^{-1})\,(x(p))$

$\left(\frac{\partial}{\partial x^a}\right)_p (x^b) = \partial_a(x^b \circ x^{-1})\; x(p)$

$= \underbrace{\partial_a(\mathrm{proj}^b)}_{\delta_a^b}(x(p))$