DIFFERENTIAL GEOMETRY LEC- 1

SIMAR NARULA

Introduction and Motivation

- In mathematics we are intersted in defining structures over set For eg vector space $(V,+,\cdot)$ group $(G,\cdot)$ and fields $(F,+,\cdot)$ are all structures. Here the structure given to the set is of a (or multiple) map. For example in case of vector space we have two maps $+ : V \times V \to V$ and $\cdot : F \times V \to V$
- Topology is also provides a structure on the set. But unlike the previously discussed structures topology is not a map but it is itself a set.
- But why Topology? We want to generalize the notion of convergence of a sequence or continuity of a map from $\mathbb{R}^n$ to a general set. Now these notions depend on "open intervals / open balls" For eg $f(x)$ is continuous at x_0 if for any x_n (in $\mathbb{R}$) lying in open interval $(x_0-\epsilon, x_0+\epsilon)$ $\exists\ \delta$ s.t. $f(x)$ lies in $(f(x_0)-\delta, f(x_0)+\delta)$ Topology generalized the idea of open intervels / open ball to open sets.

Topological Spaces

Def:
Let X be a set. A topology on X is a set $T \subseteq \mathcal{P}(X)$ such that

1. $\phi \in T$ and $X \in T$
2. If $U, V \in T$ then $U \cap V \in T$
3. If $\{U_\alpha\}_{\alpha \in A} \subseteq T$ then $\bigcup_{\alpha \in A} U_\alpha \in T$

The pair (X, T) is called topological space.

Given a topological space (X, T) a subset of X $S \subseteq X$ is said to be open if $S \in T$ and closed if $X \setminus S \in T$

So open & closed set are defined w.r.t. a topology.

In general, a set $S \subseteq X$ can be

1. both open & closed
2. open but not closed
3. closed but not open
4 neither open nor closed.

Eg Let $X = \{a, b, c\}$.

$T_1 = \{\phi, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}\}$ Yes

$T_2 = \{\phi, X\}$ (Chaotic Topology) Yes

$T_3 = \mathcal{P}(X)$ (Discrete Topology) Yes

$T_4 = \{\phi, \{a\}, \{b\}, \{b,c\}, \{a,c\}, \{a,b,c\}\}$ No

Standard Topology on $\mathbb{R}^n$

$$U \in T_{std, \mathbb{R}^n} :\Longleftrightarrow \forall p \in U \ \exists r \in \mathbb{R}^+ : B_r(p) \subseteq U$$

here $B_r(x) := \left\{ y \in \mathbb{R}^n \,\middle|\, \sqrt{\sum_{i=1}^{n} (y_i - x_i)^2} < r \right\}$

Claim: $(\mathbb{R}^n, T_{std, \mathbb{R}^n})$ form a topological space

Proof (H.W.)

1. $\phi \in T_{std, \mathbb{R}^n}$

as $\forall \underbrace{p \in \phi}_{\text{False}}$: ...... So statement is vacuously true

2. $M \in T_{std, \mathbb{R}^n}$

as any for any $x \in M$, $B_r(x) \subseteq M$

3. Let $U, V \in T_{std}$

Let $p \in U \cap V \Rightarrow p \in U$ and $p \in V$

$\Rightarrow \exists r, s \in \mathbb{R}^+ : B_r(p) \subseteq U \ \& \ B_s(p) \subseteq V$

Let $x = \min(r, s)$

So $B_x(p) \subseteq U \ \& \ B_x(p) \subseteq V$

$\Rightarrow B_x(p) \subseteq U \cap V$

$\therefore$ for $p \in U \cap V$ $\exists x \in \mathbb{R}^+ : B_x(p) \subseteq U \cap V$

$\Rightarrow U \cap V \in T_{std}$

4. $O = \bigcup_{\alpha \in A} U_\alpha$

Let $p \in O \Rightarrow \exists \beta \in A : p \in U_\beta$

As $U_\beta \in T_{std} \Rightarrow \exists r \in \mathbb{R}^+ : B_r(p) \subseteq U_\beta$

$\Rightarrow B_r(p) \subseteq O$

Hence $O \in T_{std}$

Continuity : (X, T_X) and (Y, T_Y) are top. space

$\phi : X \longrightarrow Y$ is a map

ϕ is continuous iff $\forall\ V \in T_Y$: $\text{preim}_\phi(V) \in X$

$\hookrightarrow \{x \in X \mid \phi(x) \in V\}$

i.e. a map is continuous if preimage of open sets is open

Eg $\phi : \underset{T_X = \mathcal{P}(X)}{X} \longrightarrow \underset{T_Y}{Y}$ ϕ Always continuous

$\phi : \underset{T_{std,n}}{\mathbb{R}^n} \longrightarrow \underset{T_{std,m}}{\mathbb{R}^m}$: This gives the our known defn of continuity

Homeomorphism $\Rightarrow$ Let $\phi : X \to Y$ be a bijection. Equip (X, T_X) & (Y, T_Y)

ϕ is a homeo iff

$\hookrightarrow$ bijective _bicontinuous

1. $\phi : X \longrightarrow Y$ is continuous
2. $\phi^{-1} : Y \longrightarrow X$ is continuous

Homeos are structure preserving maps in topology. They provide a one to one pairing of open sets of X with open sets of Y

If a homeo exist b/w (X, T_X) and (Y, T_Y) then $(X, T_X) \cong_{top} (Y, T_Y)$

Subset Topology (X, T)

Let $A \subset X$

$$T_{|A} := \{U \cap A \mid U \in T\} \subseteq P(A)$$

is induced topology on A by X

Proof that $T_{|A}$ is indeed a topology

1. $\phi \in T \Rightarrow \phi \cap A \in T_{|A} \Rightarrow \phi \in T_{|A}$
2. $X \in T \Rightarrow X \cap A \in T_{|A} \Rightarrow A \in T_{|A}$
3. $U, V \in T_{|A} \Rightarrow \exists\ \tilde{U}, \tilde{V} \in T : U = \tilde{U} \cap A$, $V = \tilde{V} \cap A$

   $\Rightarrow U \cap V = \underbrace{\tilde{U} \cap \tilde{V}}_{\tilde{W} \in T} \cap A$

   $\Rightarrow U \cap V = \tilde{W} \cap A \Rightarrow U \cap V \in T_{|A}$
4. Similar proof for unions

Convergence

A sequence (i.e. a map $q : \mathbb{N} \to X$) on a top space (X, T) is said to converge against a 'limit point' $x \in X$ if

$$\forall \underset{\substack{\Psi \\ a}}{U} \in X \quad \exists N \in \mathbb{N} : \forall n > N \quad q(n) \in U$$

for all open sets containg a (also called open neighbourhood of a)

Eg. $(M, \{\phi, M\})$ $\rightarrow$ Any sequence converges against every point.
So we see that limit point as defined above may not be unique
But in physics (to atleast have some notion of reality left) we need a unique limit point (if sequence is convergent)

So we only deal with topologies in physics which are 'Hausdorff'

Hausdorff (T2) : (X, T) is Hausdorff if for any two points there exist non intersecting open neighbourhood of these two points $\rightarrow$

$$\forall p, q \in X : p \neq q \Rightarrow \exists \underset{p\in}{U}, \underset{q\in}{V} \in T : U \cap V = \phi$$

Note that all convergent sequences in a Hausdorff topology have a single limit point
(Proof is trivial & left as an exercise to the reader)

Compactness

A topological space (X, T) is called compact if every open cover C has a finite subcover $\tilde{C}$

Open Cover : $C \subseteq T$ is called an open cover if $\bigcup C = M$

(so C is a collection of open sets)

(union of open sets in C)

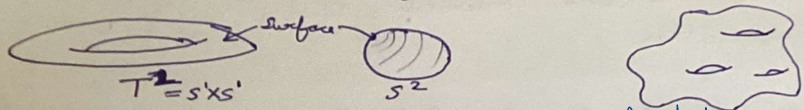
Finite subcover : is a cover $\tilde{C}$ s.t. $\underset{\text{finite}}{\tilde{C}} \subseteq C$

i.e. from a cover C take out elements so that remaining still union to M. If the remaining set is finite it is called a subcover

We will not use compactness in these lectures but it is used a lot in physics and so I have defined it here.

Topological Manifolds

→ It is a topological space that locally looks like $\mathbb{R}^d$ for some fixed d.

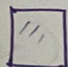


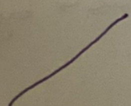
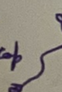
Def. A paracompact, Hausdorff topological space $(M, \mathcal{O})$ is called a d-dimensional (topological) manifold if for every point $p \in M$ $\exists$ an ~~open~~ open neighbourhood U around p (i.e. $p \in U \in \mathcal{O}$) and a homeo $x : U \longrightarrow x(U) \subseteq \mathbb{R}^d$

won't discuss this property. It is a very weak notion of compactness. Almost all the spaces you will encounter in physics are paracompact.

Note that dimension is well defined bcoz if $U \cap V \neq \emptyset$

$$\left.\begin{aligned} x &: U \longrightarrow x(U) \subseteq \mathbb{R}^d \\ y &: V \longrightarrow y(V) \subseteq \mathbb{R}^{d'} \end{aligned}\right\} \quad d = d'$$

$S^2 \cong_{top}$  $\cong_{top}$  $\not\cong$

$\mathbb{R} \cong_{top}$  $\cong_{top}$ 

Def Let $(M, \mathcal{O})$ be a top manifold of dim d. Then a pair (U, x) where $U \in \mathcal{O}$ and $x : U \to x(U) \subseteq \mathbb{R}^d$ is called a chart of the manifold

The component f^ns of x, $x^i : U \to \mathbb{R}$ $\quad x^i(p) = proj_i(x(p))$ are called coordinates of point $p \in U$ w.r.t chart (U, x)

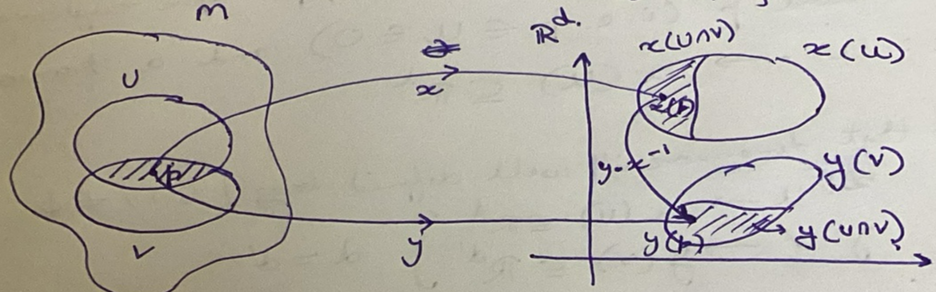
Obviously there exists a set A of charts such that $\bigcup_{(\mathcal{U},x)\in A} \mathcal{U} = M$ and there will be many ~~such~~ charts that have non zero overlap. Such a coll^n A is called an atlas.

Def Two charts $(\mathcal{U}, x)$ & (V, y) are C^0 compatible if either

(a) $U \cap V = \phi$

(b) $U \cap V \neq \phi$: $y \circ x^{-1}$ is a continuous ~~map~~ as a map from $\mathbb{R}^d$ to $\mathbb{R}^d$

(coordinate change) map or chart transition map



Note: For a topological manifold any two charts are C^0 compatible. (as x is homeo → bijective, bicontin

* A C^0 atlas is called maximal if any chart (U, x) that is C^0 compatible with any chart $(V, y) \in A$ is already contained in the atlas

Example $\rightarrow$

On Trajectories in Physics

In physics we come across maps like trajectories $\gamma : \mathbb{R} \longrightarrow M$. (Say a truck moving on surface of earth)

We require these trajectories to be continuous

Response 1 $\rightarrow$ γ is continuous as a map b/w $(\mathbb{R}, \mathcal{O}_{std}), (M, \mathcal{O})$

But we have never done this till now.

Instead what we do $\rightarrow$

We just consider a portion of M (an open subset U)

What happens if we shift our coordinate system to y

$$\begin{tikzcd}
 & & y(U) \subseteq \mathbb{R}^d \\
\mathbb{R} \supseteq \text{preim}(U) \arrow[rru, "y \circ \gamma"] \arrow[rr, "\gamma"] \arrow[rrd, "x \circ \gamma"'] & & U \subseteq M \arrow[u, "y"'] \arrow[d, "x"] \\
 & & x(U) \subseteq \mathbb{R}^d \arrow[uu, bend right, "y \circ x^{-1}"']
\end{tikzcd}$$

we talk about continuity of this map as a map from $\mathbb{R} \rightarrow \mathbb{R}^d$

(i.e. we chose a coordinate system first)

But why is this justified? It looks like here our physical reality is coordinate dependent

Now see, $$y \circ \gamma = \underbrace{(y \circ x^{-1})}_{\substack{\text{coordinate} \\ \text{change map which} \\ \text{are continuous for top. manifolds}}} \circ \underbrace{(x \circ \gamma)}_{\text{map on which we discussed continuity}}$$

As composition of two continuous maps is continuous. We see that $y \circ \gamma$ is also continuous & so our physical reality is not coordinate dependent (i.e. we could choose any coordinate map x & discuss continuity on $x \circ \gamma$ and due to chart changing map being continuous, any $y \circ \gamma$ will also be continuous)

But now if we want to discuss about differentiability we cannot do so without discussing about differentiability on $y \circ x^{-1}$ which is not implied by our structure of top. manifold. And if we ignore differentiability on coordinate change map & just talk about differentiability on $x \circ \gamma$ then our physical reality will become coordinate dependent

More on this in the next lecture