- A rep. $D(g)$ is completely reducible if it is equivalent to a representation of the following form

$$\begin{pmatrix} D_1(g) & 0 & 0 \\ 0 & D_2(g) & \\ 0 & 0 & D_3(g) \\ & & & \ddots \end{pmatrix} = \text{diag}\,(D_1(g), D_2(g), D_3(g), \ldots)$$

where each $D_j(g)$ is an irrep.

Represented as

$$\bigoplus_{j=1}^{N} D_j(g).$$

Nice result: Any representation of a finite group is completely reducible.

We now prove the 1st so called "nice" result.

Lemma Every repr of a finite group is equivalent to a unitary rep.

P.T.O.

Pf Suppose $D(g)$ is a rep of G.

$$S = \sum_{g \in G} D^{\dagger}(g) D(g).$$

See that $S^{\dagger} = S$ & $\langle v|S|v\rangle = \sum_{g\in G} \|D(g)\vec{v}\|^2 \geq 0$.

& actually > 0 as $D(e) = \mathbb{1}$

So, S is psd.

We now use the most used theorem in all of QM i.e. the spectral theorem.

As S is Hermitian, $\exists\, U$ s.t.

$$S = U^{\dagger} d U \text{ where } d = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$= \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3, \ldots)$

where λ_i are the eigenvalues of S which are ≥ 0, as S is psd.

Claim $\lambda_i > 0$.

$\exists\, \vec{v_i}$ s.t.

$$\langle v_i|S|v_i\rangle = \lambda_i |v_i|^2$$
$$\Uparrow$$
or $$S|v_i\rangle = \lambda|v_i\rangle$$

$(X^{-1})^{\dagger} = X^{-1}$

We just saw,

$$\langle v|S|v\rangle > 0 \Rightarrow \lambda_i > 0.$$

Then, define $X = S^{1/2} \equiv U^{-1} \begin{pmatrix} \sqrt{\lambda_1} & & \\ & \sqrt{\lambda_2} & \\ & & \ddots \\ & & & \sqrt{\lambda_n} \end{pmatrix} U = U^{\dagger} d^{1/2} U$

$X^{\dagger} = U^{\dagger} d^{1/2} U$

$X^{\dagger}X = U^{\dagger} d^{1/2} U U^{\dagger} d^{1/2} U$

$= U^{\dagger} d U = S$

Let $D'(g) = X D X^{-1}$

$D'^{\dagger} D' = (X^{-1})^{\dagger} D^{\dagger} X^{\dagger} X D X^{-1}$

$= (X^{-1})^{\dagger} D^{\dagger} S D X^{-1}$

$= X^{-1} \cancel{S} X^{-1} = I$.

$D^{\dagger} S D = D^{\dagger}(g) \sum_h (D^{\dagger}(h) D(h)) D(g)$

$= \sum_h D^{\dagger}(hg) D(hg)$

$= \sum_{a\in G} D^{\dagger}(a) D(a) = S$

$= X^2$

The

ie "2nd" nice result —

Every representation of a finite group is completely reducible.

Pf By similarity argument & the previous result, enough to choose unitary rep.

If it is irreducible, we are done.

Suppose not. Then, $\exists P$ s.t. → Projector is chosen s.t. $P^\dagger = P$ (w.l.g)

$$PDP = DP \quad \forall g \in G$$ (we won't mention this from now on)

$$\Rightarrow PD^\dagger P = PD^\dagger \qquad (D^\dagger = D^{-1}) \ (D \text{ is unitary})$$

$$\Rightarrow PD^{-1}(g)P = PD^{-1}(g)$$

$$\Rightarrow PD(g^{-1})P = PD(g^{-1}) \quad \forall g \in G$$

$$\Rightarrow PDP = PD \quad \forall g \in G$$

Is $I-P$ an invariant subspace?

$$(I-P)D(I-P) = D - \cancel{PD} - DP + \cancel{PDP}$$

$$= D - DP = D(I-P).$$

Thus, both P & $I-P$ are invariant subspaces.

Basically,

$$\begin{bmatrix} PP & 0 \\ 0 & (I-P)(I-P) \end{bmatrix}$$

there is no coupling. Now, we can keep reducing the matrix rep. in each of these subspaces until we get to irreps.

Thus, any finite group rep. is completely reducible.

Now, we come to the high point of this lecture, i.e. Schur's lemma.

P.T.O.

Part 1

If $D_1(g) A = A D_2(g)$ where D_1 and D_2 are inequivalent irreps, then $A = 0$.
$\forall g \in G$

Pf Let P be a projector s.t.
$AP = 0$

$\Leftrightarrow \exists |u\rangle$ s.t. $A|u\rangle = 0$.

$D_1 A P = A D_2 P = 0 \; \forall g \in G$ Say $P = |u\rangle\langle u|$ w.l.g.

$\Rightarrow \underline{A D_2 |u\rangle} \langle u| = 0 \; \forall g \in G$.

As D_2 is an irrep, $D_2|u\rangle$ goes through the whole space of vectors.

The basis can be constructed out of $D_2|u\rangle \; \forall g$ linear combination.

$\Rightarrow A|i\rangle = 0 \; \forall i = 1, \ldots, N$.

$\Rightarrow \boxed{A = 0}$

A similar argument goes if $\langle u|A = 0$ for some $\langle u|$.

Now, suppose $\nexists |u\rangle$ or $\langle u|$ s.t. the following happens

That means :-

i) A is square.

ii) A is invertible

$\Rightarrow A^{-1} D_1 A = D_2$ $(\Rightarrow\Leftarrow)$.

Thus, we have proved this result

Part 2

If $D(g) A = A D(g) \; \forall g \in G$, $A = \lambda I$.

Pf It is clear here that A is square. Now, either A is invertible or not. If not, $A = 0 = 0 I$. Else, $\exists \lambda \in \mathbb{R}$ s.t. $|A - \lambda I| = 0$

$\Rightarrow$ $A - \lambda I$ is not invertible.

$$D(g)A - \lambda I D(g) = D(g)(A - \lambda I) = AD(g) - \lambda D(g)$$
$$= (A - \lambda I) D(g)$$

& $A - \lambda I$ is non-invertible

$$\Rightarrow A = \lambda I \Rightarrow A \propto I.$$

These proofs can be done in the terms of modules. The matrices form a ring which act on the ~~vector space~~/module. In this language, irreducibility $\Rightarrow$ $\boxed{\nexists \text{ proper submodule } W \text{ of } U \text{ under the irrep transform.}}$

$D(g)$ has block diagonal form.

Now, suppose $D(g)$ has block diagonal form.
The orthonormal basis states can be written as $|a, j, x\rangle$

irrep

state within irrep

other physical parameters. (how many times a is appearing)

These satisfy

$$\langle b, k, y | a, j, x\rangle = \delta_{ba}\delta_{kj}\delta_{yx}$$

We assume whenever irrep a appears, it has the same rep $D_a(g)$, w.l.g.

Under symmetry transformation,

$$|\mu\rangle \to D(g)|\mu\rangle, \quad \langle\mu| \to \langle\mu| D^\dagger(g)$$

$$O \to D(g) O D^\dagger(g)$$

If O is invariant under symmetry,

$$DO = OD \Rightarrow [O, D] = 0$$

$$\Rightarrow \langle a, j, x | [O, D] | b\, k\, y\rangle = 0$$

This is ~~required~~ enough as $D(g)$ does not move out of $|b, k, y\rangle$ or $|a, j, x\rangle$.

$$\Rightarrow \sum_{k'} \langle a, j, x | O | b, k', y\rangle \langle b, k', y | D(g) | b, k, y\rangle$$
$$- \sum_{j'} \langle a, j, x | D(g) | a, j', x\rangle \langle a, j', x | O | b, k, y\rangle = 0$$

$$= \sum_{k'} \langle a, j, x | O | b, k', y \rangle [D_b(g)]_{k'k}$$

$$- \sum_{j'} [D_a(g)]_{jj'} \langle a, j', x | O | b, k, y \rangle$$

$$\langle a, j, x | O | b, k, y \rangle \equiv O^{ab, xy}_{jk}$$

$$= \sum_{k'} O^{ab,xy}_{jk'} [D_b(g)]_{k'k} - \sum_{j'} [D_a(g)]_{jj'} O^{ab,xy}_{j'k}$$

$$= \left[O^{ab,xy} D_b - D_a O^{ab,xy} \right]_{jk} = 0$$

$$\Rightarrow O^{ab,xy} D_b = D_a O^{ab,xy}$$

If $a \neq b$, we know,

$$O^{ab,xy} = 0.$$

If not, i.e. $a = b$,

$$O^{ab,xy} \propto I$$

$$\Rightarrow (O^{ab,xy})_{jk} = f_a(x,y)\, \delta_{ab}\, \delta_{jk}$$

So observables invariant under symmetry have fixed values dependent only on the corresponding irreps.

<u>Orthogonality relations</u>

$$A^{ab}_{jl} \equiv \sum_{g \in G} D_a(g^{-1}) |a, j\rangle\langle b, l| D_b(g)$$

$$D_a(g_1)\, A^{ab}_{jl} = \sum_{g \in G} D_a(g_1 g^{-1})\, |a,j\rangle\langle b,l|\, D_b(g)$$

$$= \sum_{\substack{g_1 g^{-1} \in G \\ = k}} D_a(g_1 g^{-1})\, |a,j\rangle\langle b,l|\, D_b((g_1 g^{-1})^{-1} g_1)$$

$$= \sum_{k \in G} D_a(k)\, |a,j\rangle\langle b,l|\, D_b(k)^{-1}\, D_b(g_1)$$

$$= A^{ab}_{jl}\, D_b(g_1)$$

Schur's Lemma says $A^{ab}_{jl} = 0$ if D_a & D_b are diff.,

if they are the same, $A^{ab}_{jl} = \lambda^{ab}_{jl} I$.

$$\mathrm{Tr}(A^{ab}_{jl}) = \delta_{ab}\, \mathrm{Tr}(\lambda^{a}_{jl} I) = \delta_{ab}\, \lambda^{a}_{jl}\, n_a.$$

$$\mathrm{Tr}(A^{ab}_{jl}) = \delta_{ab} \sum_{g \in G} \langle a,l|\, D_a(g) D_a(g^{-1})\, |a,j\rangle$$

$$= \delta_{ab}\, N \langle a,l|a,j\rangle$$

$$= \delta_{ab}\, \delta_{jl}\, N$$

$$\Rightarrow \lambda^{a}_{jl} = \frac{N}{n_a}\, \delta_{jl}. \quad \Rightarrow \sum_{g \in G} D_a(g^{-1})\, |a,j\rangle\langle b,l|\, D_b(g) = \frac{N}{n_a}\, \delta_{ab}\, \delta_{jl}\, I$$

$$\Rightarrow \sum_{g\in G} \frac{n_a}{N} \langle a,k|D_a(g^{-1})|a,j\rangle \langle b,l|D_b(g)|b,m\rangle = \delta_{ab}\,\delta_{km}\,\delta_{jl}$$

$$\Rightarrow \sum_{g\in G} \frac{n_a}{N} \left[D_a(g^{-1})\right]_{kj} \left[D_b(g)\right]_{lm} = \delta_{ab}\,\delta_{km}\,\delta_{jl}$$

$$\left[D_a(g^{-1})\right]_{kj} \rightarrow \left[D_a^{\dagger}(g)\right]_{kj} = \left[D_a(g)\right]^{*}_{jk}$$

$\Rightarrow \left[\sqrt{\frac{n_a}{N}}\, D_a(g)\right]_{jk}$ are orthonormal functions of the group elements, g.

$\hookrightarrow$ The matrix elements of the unitary irreps of G are a complete orthonormal set (COS) for the vector space of the regular representation.

$N = \sum n_i^2$ — N: order of group; n_i: size of irreps (V.V.I.)

Characters

$$\chi_D(g) \equiv \mathrm{Tr}\left[D(g)\right] = \sum_i \left[D(g)\right]_{ii}.$$

$$\sum_{\substack{g\in G\\ j=k\\ l=m}} \frac{1}{N} \left[D_a(g)\right]^{*}_{jk} \left[D_b(g)\right]_{lm} = \sum_{\substack{j=k\\ l=m}} \frac{1}{n_a}\, \delta_{ab}\,\delta_{jl}\,\delta_{km}$$

$$= \delta_{ab}$$

$$\Rightarrow \frac{1}{N} \sum_{g\in G} \chi^{*}_{D_a}(g)\, \chi_{D_b}(g) = \delta_{ab}.$$

Characters of different irreps are orthogonal.

Characters are constant on conjugacy classes.

$$\hookrightarrow \mathrm{Tr}\, D(g^{-1} g_1 g) = \mathrm{Tr}\left(D(g)\,D(g^{-1})\,D(g_1)\right)$$
$$= \mathrm{Tr}\left(D(g_1)\right).$$

Characters then also form a basis for functions on the conjugacy classes.

Suppose $F(g_1)$ is such a fn.

$$F(g_1) = \sum_{a,j,k} c^a_{jk} [D_a(g_1)]_{jk}$$

F is constant on conjugacy classes

$$= \frac{1}{N} \sum_{g \in G} F(g^{-1} g_1 g)$$

$$= \frac{1}{N} \sum_{a,j,k,g} c^a_{jk} [D_a(g^{-1} g_1 g)]_{jk}$$

$$\Rightarrow F(g_1) = \frac{1}{N} \sum_{\substack{a,j,k \\ g,l,m}} c^a_{jk} (D_a(g^{-1}))_{jl} (D_a(g_1))_{lm} (D_a(g))_{mk}$$

$$= \frac{1}{n_a} \sum_{\substack{a,j,k \\ l,m}} c^a_{jk} \delta_{jk} \delta_{lm} [D_a(g_1)]_{lm}$$

$$F(g_1) = \frac{1}{n_a} \sum_{a,j,l} c^a_{jj} [D_a(g_1)]_{ll}$$

$$= \sum_{a,j} \frac{1}{n_a} c^a_{jj} \chi_a(g_1)$$

$\Rightarrow$ No. of irreps = no. of conjugacy classes.

Writing character tables. $\longrightarrow$

$$V_{\alpha a} \equiv \sqrt{\frac{k_\alpha}{N}} \chi_{D_a}(g_\alpha)$$

g_α is in the conjugacy class α, with k_α elements.

$$V^\dagger V = 1 \Leftrightarrow \frac{1}{N} \sum_{g \in G} \chi_{D_a}^*(g) \chi_{D_b}(g) = \delta_{ab}$$

$$VV^\dagger = \mathbb{1} \Rightarrow \sum_a \chi_{D_a}(g_\alpha)^* \chi_{D_a}(g_\beta) = \frac{N}{k_\alpha} \delta_{\alpha\beta}$$

Suppose $D(g)$ contains

$$\sum_a \oplus D_a$$

Then $$\frac{1}{N}\sum_{g\in G} \chi^*_{D_a}(g)\chi_D(g) = m_a^D$$

($\to$ no. of times irrep a appears in $D(g)$)

$$\Downarrow$$

For $D = R$ (regular representation),

$\chi_A(e) = n_a$.

Constructing the character table

$$\sum_{g\in G} \chi^*_{D_a}(g)\,\chi_{D_b}(g) = N\delta_{ab}$$

$$\sum_a \chi^*_{D_a}(g_\alpha)\,\chi_{D_a}(g_\beta) = \frac{N}{k_\alpha}\delta_{\alpha\beta}$$

$k_\alpha \to$ order of that conjugacy class.

See that χ's have bidirectional orthogonality, and are thus easy to construct using orthogonality arguments.

Suppose, let us try to make the character table for S_3.

We know characters are constant over conjugacy classes

So, a typical character table looks like this:-

	e	a_1, a_2	a_1, a_2, a_3
T	1	1	1
P	1		
R	2		

We know,

$\sum n_i^2 = N$

$\Rightarrow \sum_{i=1}^{3} n_i^2 = 6$

only solution is $1, 1, 2$.

$\downarrow$

trivial

Now, let us think of a 1D rep which satisfies the requirements of a representation.

$$S_3 = \{e, (12), (23), (13), (123), (132)\}$$

square to e

so, ± 1 for all three

So, if 1D, for it to be a distinct rep, must be -1.

multiply to give rotations

Satisfies this

square to each other, multiply to give e

same value for both

i.e. $a^3 = e$

$\Rightarrow 1$.

But what is this?

This is the parity of the permutation i.e. reflections are odd permutations, thus have $-$ parity. Rotations are even permutations here, so $+1$.

(All 1D reps are irreps) (excluding scalar factor isomorphisms)

Currently, the character table looks like:-

a_3, a_4, a_5 are reflections

	e	a_1, a_2 (rotations)	a_3, a_4, a_5
T	1	1	1
P	1	1	-1
R	2	a $=(-1)$	b $(=0)$

$\to$ 2×2 matrix

Now we use the orthogonality relations to get

$$\sum_{g \in G} \chi_{D_a}^{*}(g)\, \chi_{D_b}(g) = N\,\delta_{ab}$$

$$2 + 2a + 3b = 0$$

$$\Rightarrow \quad 2 + 2a + 3b = 0$$

$$\Rightarrow b = 0 \ \& \ a = -1.$$

Try it out for D_8 based on symmetry arguments.

Remember every group based on permutations has parity reps. & every group has trivial rep.

$$\sum n_i^2 = 8 \quad \sum_{i=3}^{5} n_i^2 = 6 \Rightarrow 1, 1, 2$$

We already two of them are trivial & parity, both 1D.

To get an idea of the other irreps, let us look at other squaring squaring conjugacy classes.

	e	r, r^3	$r^2 s, s$	r^2	sr, sr^3
T	1	1	1	1	1
P	1	1	-1	1	-1
$\mathbb{1}_A$	1	-1	1	1	-1
$\mathbb{1}_B$	1	-1	-1	1	1
$\mathbb{2}$	2	a	b	c	d

The elements are

e, r, r^2, r^3

s, sr, sr^2, sr^3

Use the fact that

$r^2 s = r s r^{-1}$

$r = s r^{-1} s^{-1}$

$r = s r^3 s^{-1}$

$s r^{-2} = s r^2$

r^2 commutes with everything

Now, we again use orthogonality relations, but now columnwise.

$$\sum_a \chi^*_{D_a}(g_\alpha)\, \chi_{D_a}(g_\beta) = \frac{N}{k_\alpha} \delta_{\alpha\beta}$$

$$1.1 + 1.1 + 1.-1 + 1.-1 + 2a = 0$$

$\Rightarrow$

$\Rightarrow$ $a = 0$, similarly $b = d = 0$

$\Rightarrow c = -2$

See how fun it is to do this if you have free time.

The primary concern

Lie groups $\longrightarrow$ Abstract formalism of Lie groups

Physicists use only the representation - theoretic aspects.

Suppose our group elements $g \in G$ depend smoothly on a set of continuous parameters, $g(\alpha)$.

Most convenient — set $g(\alpha)\big|_{\alpha=0} = e$. (we always need an identity).

$$D(\alpha)\big|_{\alpha=0} = I$$

Lie groups are "good" structures — they can be mapped to smooth manifolds.

Taylor expand around identity.

$$D(d\alpha) = 1 + i\, d\alpha_a X_a + \cdots$$

Before that,

$$e^A = ? \implies e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

$$X_a \equiv -i \frac{\partial}{\partial \alpha_a} (D(\alpha))\Big|_{\alpha=0}$$

X_a, $a = 1$ to N are called the generators of the group.

Groups represented by $D(\alpha)$ are called Lie groups.

Using this notation,

$$D(\alpha) = \lim_{k\to\infty} \left(1 + i\frac{\alpha_a X_a}{k}\right)^k = e^{i\alpha_a X_a}.$$

This is the exponential parametrization of a Lie group.

The actual definition of the Lie group is as a finite dimensional real smooth manifold. we'll work with the expinential parametrization instead.

Now,

$$U(\lambda) = e^{i\lambda\alpha_a X_a} \quad (\alpha_a X_a \text{ is fixed}).$$

But as the group has multiple generators,

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} \neq e^{i(\alpha_a+\beta_a)X_a}.$$

However, if they form a group,

$$e^{i\alpha_a X_a} e^{i\beta_b X_b} = e^{i\delta_a X_a}$$

Let us try to find δ_a is. This will give us the most important property of Lie groups.

However, we'll do it in a slightly different way.

$$i\delta_a X_a = \ln\left(1 + e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1\right)$$

$$= \left(e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1\right) - \frac{\left(e^{i\alpha_a X_a} e^{i\beta_b X_b} - 1\right)^2}{2!} + \cdots$$

$$= \left(i\alpha_a X_a + i\beta_b X_b - \alpha_a X_a \beta_b X_b - \frac{(\alpha_a X_a)^2}{2} - \frac{(\beta_b X_b)^2}{2}\right) - \left(i\alpha_a X_a + i\beta_b X_b\right)^2 + O(X_\alpha^3) + \cdots$$

$$= i(\alpha_a + \beta_a)X_a - \alpha_a X_a \beta_b X_b - \frac{(\alpha_a X_a)^2}{2} - \frac{(\beta_b X_b)^2}{2}$$

$$= i(\alpha_a + \beta_a)X_a - \frac{1}{2}[\alpha_a X_a, \beta_b X_b] + O(X^3) + \left((\alpha_a X_a)^2 + (\beta_b X_b)^2 + \alpha_a \beta_b X_a X_b + \beta_a \alpha_b X_a X_b\right.$$

$$[\alpha_a X_a, \beta_b X_b] = -2i(\delta_c - \alpha_c - \beta_c) X_c + \ldots = i\nu_c X_c$$

Should be for δ_a to exist.

$$\Rightarrow \alpha_a \beta_b [X_a, X_b] = i\nu_c X_c$$

This is true for all α, β,

$$\nu_c = f_{abc} \alpha_a \beta_b$$

$$\boxed{[X_a, X_b] = i f_{abc} X_c}$$

structure constant

This is enough to characterize the whole group

$$f_{abc} = -f_{bac} \quad \text{(Easy to see!)}$$

However, remember that we did this only till the 2^{nd} order. So, is $[X_a, X_b] = i f_{abc} X_c$ enough? Turns out it is, miraculously.

<u>Jacobi identity</u>: The generator follow the following property:-

$$[X_a, [X_b, X_c]] + \text{cyclic permutations} = 0$$

True for all $N \times N$ matrices. So, does not add much information to the Lie group.

Now, curiously, if one can find generators such that they have the same structure constant, they are the same group upto isomorphism.

We will not delve much into the abstract notions of Lie groups.

Quick note:-

<u>Def.</u> $[T_a]_{bc} \equiv -i f_{abc}$.

Then, $[T_a, T_b] = i f_{abc} T_c$

Structure constants themselves form a representation of the Lie group, called the adjoint representation.

Now, let us come to some calculations, through examples of Lie groups. Let us see how we calculate generators of Lie groups

The most known Lie group, SU(2)

These are 2X2 matrices on $\mathbb{C}$ which satisfy

$$U^\dagger U = U U^\dagger = I \quad \& \det U = I$$

First, let us parametrize it close to identity.

i is taken to have Hermitian generators

$$U = (\mathbb{1} + i\varepsilon)$$

$$U^\dagger = (\mathbb{1} - i\varepsilon^\dagger)$$

$$U U^\dagger = (\mathbb{1} + i\varepsilon)(\mathbb{1} - i\varepsilon^\dagger)$$

$$= \mathbb{1} + i(\varepsilon - \varepsilon^\dagger) + O(\varepsilon\varepsilon^\dagger)$$

$$\det(\mathbb{1} + i\varepsilon) \approx Tr(\varepsilon) = 0.$$

At first order, $\varepsilon = \varepsilon^\dagger$.

So, the generators are Hermitian.

We choose them to be traceless as well, and out come the Pauli matrices.

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Remember: SU(2) can have higher dimensional representation

Explore different Lie groups

1. $SU(N) \to N \times N$ dimensional unitary matrices with $\det = 1$

   We use the same derivation to conclude that the generators are $N \times N$ traceless Hermitian matrices.

2. $SO(N)$

3. Lorentz group.

4. Poincare group.

5. $SL(2, C)$ → special linear transformations i.e. $\det = 1$.

6. $SO(3,1)$

$Sp(n) \to D^T E D = E$

A look at classical mechanics

$$\{a, b\}_{p,q} = \sum_i \left[\frac{\partial a}{\partial q_i} \frac{\partial b}{\partial p_i} - \frac{\partial a}{\partial p_i} \frac{\partial b}{\partial q_i} \right]$$

$$= F^T E G$$

where $E = \begin{bmatrix} 0 & (1)_{n\times n} \\ (-1)_{n\times n} & 0 \end{bmatrix}$

$$F = \begin{pmatrix} \frac{\partial a}{\partial q_1} \\ \vdots \\ \frac{\partial a}{\partial q_n} \\ \frac{\partial a}{\partial p_1} \\ \vdots \\ \frac{\partial a}{\partial p_n} \end{pmatrix} \qquad G = \begin{pmatrix} \frac{\partial b}{\partial q_1} \\ \vdots \\ \frac{\partial b}{\partial q_n} \\ \frac{\partial b}{\partial p_1} \\ \vdots \\ \frac{\partial b}{\partial p_n} \end{pmatrix}$$

Invariance of Poisson brackets becomes :-

$$F^T E G = (F')^T E G'$$

in the primed coordinates