⊙ Dual space and bra :

Suppose we are given a vector space V defined over a field F ($\mathbb{R}$ or $\mathbb{C}$). Take all the linear operators $\Omega : V \to F$. The collection of all such Ω's satisfy the vector space axioms. So, this collection of all Ω's such that $\Omega : V \to F$ itself forms a vector space. It is called the <u>Dual space</u> of V and denoted by V^*.

Suppose we have a vector field V defined over $\mathbb{C}$. Take any vector, $|v\rangle$, from it. It has an associated 'dual' or 'bra' with it, defined as the operator $\Omega : V \to \mathbb{C}$ s.t.

$$\Omega |f\rangle = \langle v|f\rangle \quad \forall \ |f\rangle \in V.$$

[$\langle v|f\rangle$ is the inner product of $|v\rangle$ and $|f\rangle$]

Clearly, Ω is an element of V^*.

Ω is also denoted as $\langle v|$.

V^* is also called the corresponding <u>bra space</u> of the <u>vector or 'ket' space V</u>.

This bra and ket formalism is due to Dirac.

Basis for Dual space :-

Consider that our vector space V has an orthonormal basis $\{|e_1\rangle, |e_2\rangle, \ldots, |e_n\rangle\}$. We claim that set of its duals, i.e. $\{\langle e_1|, \langle e_2|, \ldots, \langle e_n|\}$ provide a basis for V^*.

Pf:- Suppose $\Omega \in V^*$ and so $\Omega : V \to \mathbb{C}$.

Take any vector $|v\rangle \in V$ and suppose

$$|v\rangle = \sum_i v_i |e_i\rangle$$

$$\Omega|v\rangle = \sum_i \Omega v_i |e_i\rangle$$
$$= \sum_i v_i (\Omega|e_i\rangle)$$

Since our choice of v_i's are arbitrary, the action of Ω on $|v\rangle$ depends on the action of Ω on each $|e_i\rangle$, $i = 1, 2, \ldots, n$.

Suppose $\Omega|e_i\rangle = a_i$, where $a_i \in \mathbb{C}$ for $i = 1, 2, \ldots, n$.

Then, $\Omega = \sum_j a_j \langle e_j|$, because

$$\Omega|e_i\rangle = \sum_j a_j \langle e_j|e_i\rangle$$
$$= \sum_j a_j \delta_{ij}$$
$$= a_i$$

~~Thus, $\{\langle e_1|, \langle e_2|, \ldots, \langle e_n|\}$ span V^* and sufficient for it. So, it provides a basis for V^*.~~

Matrix representation of bra:

Suppose $\alpha|v\rangle = |\alpha v\rangle$ $[\alpha \in \mathbb{C}]$

Then, $\langle \alpha v| = \langle v|\alpha^*$.

$\boxed{Pf:}$ $\langle g|\alpha v\rangle = \alpha\langle g|v\rangle$

$\Rightarrow \langle \alpha v|g\rangle = \alpha^*\langle v|g\rangle$ [Taking conjugates)

$\Rightarrow (\langle \alpha v|)\,|g\rangle = (\langle v|\alpha^*)\,|g\rangle \quad \forall\, |g\rangle \in V.$

$\therefore \langle \alpha v| = \langle v|\alpha^*$. [Proved]

Now, suppose $|v\rangle = \sum v_i|e_i\rangle$ and $|f\rangle = \sum f_i|e_i\rangle$

clearly, $\langle v| = \sum \langle e_i|\, v_i^*$.

And, $\langle v|f\rangle = \sum v_i^* f_i$

$$= \begin{pmatrix} v_1^* & v_2^* & \cdots & v_n^* \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$ is column matrix representation of $|f\rangle$,

so $\begin{pmatrix} v_1^* & v_2^* & \cdots & v_n^* \end{pmatrix}$ is the row matrix representation of $\langle v|$.

So, $\langle v|$ is the <u>transpose conjugate</u> of $|v\rangle$.

Key take-away:

We can view inner product as a 'bra' acting on a 'ket'.

⊙ Bra followed by ket gives a scalar (inner product).

⊙ Ket followed by bra gives an operator (outer product).

$$|v\rangle\langle w| = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \begin{pmatrix} w_1^* & w_2^* & \cdots & w_n^* \end{pmatrix}$$

$$= \begin{pmatrix} v_1 w_1^* & v_1 w_2^* & \cdots & v_1 w_n^* \\ v_2 w_1^* & \ddots & & \\ \vdots & & & \\ v_n w_1^* & \cdots & & v_n w_n^* \end{pmatrix}.$$

⊙ Note:-

$$|v\rangle = \sum_i |e_i\rangle\langle e_i|v\rangle$$

$$= \left(\sum_i |e_i\rangle\langle e_i|\right)|v\rangle$$

clearly, $I = \sum_i |e_i\rangle\langle e_i|$ (Identity)

The operator $P_i = |e_i\rangle\langle e_i|$ is the ith Projector operator.

Thus, $(AB)_{ij} = \langle e_i|AB|e_j\rangle = \langle e_i|AIB|e_j\rangle$

$$= \langle e_i|A\left(\sum_k |e_k\rangle\langle e_k|\right)B|e_j\rangle$$

$$= \sum_k \langle e_i|A|e_k\rangle\langle e_k|B|e_j\rangle$$

Adjoint:

If $\Omega|v\rangle = |\Omega v\rangle$ then the operator that transforms $\langle v|$ to $\langle \Omega v|$ is Ω^{+} (adjoint of Ω).

$$\langle \Omega v| = \langle v|\Omega^{+} \quad \text{(NOTE the order)}$$

Ex: Matrix corresponding to Ω^{+} is the transpose conjugate of Ω, i.e. $(\Omega^{+})_{ij} = \Omega_{ji}^{*}$.

[HINT:- Expand everything in terms of matrix]

Ex: $(\Omega\Lambda)^{+} = \Lambda^{+}\Omega^{+}$

Hermitian and Anti-hermitian operator:

- An operator Ω is hermitian if $\Omega = \Omega^{+}$
- An operator Ω is anti-hermitian if $\Omega = -\Omega^{+}$.

Unitary operator:

- An operator U is unitary if $UU^{+} = U^{+}U = I$.

Thm: An operator U is unitary iff it preserves inner product.

Pf: Suppose U acts on all $|v\rangle$ of V.

$$\langle v_1|v_2\rangle = \langle Uv_1|Uv_2\rangle$$
$$= \langle v_1|U^{+}U|v_2\rangle.$$

True for all $|v_1\rangle, |v_2\rangle \in V$ iff $U^{+}U = I$.

[Proved]

Thm:- If A is a transformation matrix of orthonormal bases then A is unitary.

Pf:

$$\{|e_1\rangle, \dots, |e_n\rangle\} \xrightarrow{A} \{|f_1\rangle, \dots, |f_n\rangle\}.$$

$$|f_i\rangle = \sum_{j=1}^{n} A_{ij} |e_j\rangle$$

$$\langle f_k| = \sum_{l=1}^{n} \langle e_l| A_{lk}^*$$

$$\langle f_k|f_i\rangle = \sum_{l=1}^{n} \sum_{j=1}^{n} \langle e_l| A_{lk}^* A_{ij} |e_j\rangle$$

$$\Rightarrow \delta_{ki} = \sum_{l=1}^{n} \sum_{j=1}^{n} A_{lk}^* A_{ij}\, \delta_{lj}$$

$$\Rightarrow \delta_{ik} = \sum_{j=1}^{n} A_{ij} A_{jk}^*$$

$\Rightarrow AA^{\dagger} = I$. Similarly, $A^{\dagger}A = I$. Thus, A is unitary.

[Proved]

Normal operator:

N is normal if $NN^{\dagger} = N^{\dagger}N$.

Clearly, ~~N is normal~~.

Hermitian, Anti-hermitian and unitary operators are normal operators.

$\boxed{\text{Basis transformation:}}$

Suppose we transform from one orthonormal basis $\{|e_1\rangle, \dots, |e_n\rangle\}$ to $\{|f_1\rangle, \dots, |f_n\rangle\}$ using the unitary operator A.

$$\text{Now, } |f_i\rangle = \sum_{j=1}^{n} A_{ij} |e_j\rangle \quad \text{——— (1)}$$

Suppose a vector $|v\rangle$ is expressed in the earlier basis $\{|e_1\rangle, \dots, |e_n\rangle\}$ and the transformed basis $\{|f_1\rangle, \dots, |f_n\rangle\}$ as follows:

$$|v\rangle = \sum_{i=1}^{n} v_i |e_i\rangle \Rightarrow |v\rangle = \sum_{j=1}^{n} v_j |e_j\rangle$$

$$|v\rangle = \sum_{i=1}^{n} v_i' |f_i\rangle$$

What is the relation between the old and transformed coordinates?

$$\text{Now, } \sum_{j=1}^{n} v_j |e_j\rangle = \sum_{i=1}^{n} v_i' |f_i\rangle$$

$$\Rightarrow \sum_{j=1}^{n} v_j |e_j\rangle = \sum_{i=1}^{n} v_i' \left(\sum_{j=1}^{n} A_{ij} |e_j\rangle \right)$$

$$\Rightarrow \sum_{j=1}^{n} \left(v_j - \sum_{i=1}^{n} A_{ij} v_i' \right) |e_j\rangle = 0$$

Since the basis in orthonormal, so each term in bracket is zero.

$$\therefore \quad v_j = \sum_{i=1}^{n} A_{ij} v_i'$$

So, if $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$, $v' = \begin{pmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{pmatrix}$, then $v = A v'$

$$\Rightarrow A^{\dagger} v = A^{\dagger} A v'$$

$$\Rightarrow \boxed{v' = A^{\dagger} v} \quad \left(\because A \text{ is unitary, so } AA^{\dagger} = I \right)$$

What about transformation of operator matrix?

Suppose $\Omega : V \to V$.

Let $\Omega|v\rangle = |w\rangle$

Now, suppose in old basis representation, the matrix of Ω is Ω.

Then, $\Omega V = W$, where V and w are the column matrices of old coordinates.

If modified coordinates of the vector are the column matrices are V' and W' then,

$$V' = A^{\dagger} V \Rightarrow V = AV'$$
$$W' = A^{\dagger} W \Rightarrow W = AW'$$

$$\therefore \Omega(AV') = AW'$$
$$\Rightarrow W' = (A^{\dagger}\Omega A)V' \quad [A \text{ is unitary, so } A^{\dagger} = A^{-1}]$$

Thus, matrix changes from Ω to $A^{\dagger}\Omega A$.

WORD OF CAUTION

The vectors and operators are abstract objects that have a particular representation in a given orthonormal basis. The representation changes when we shift the basis. The abstract objects remain the same.

Eigenvalues and Eigenvectors:-

If $\Omega|v\rangle = \omega|v\rangle$ for some $|v\rangle \in V$ ($|v\rangle \neq |0\rangle$) and scalar ω, ω is called an eigenvalue and $|v\rangle$ the corresponding eigenvector.

If α is an arbitrary scalar,

$$\Omega(\alpha|v\rangle) = \alpha\,\Omega|v\rangle = \alpha\,\omega|v\rangle = \omega(\alpha|v\rangle)$$

Thus, $\alpha|v\rangle$ is also a corresponding eigenvector.

All eigenvectors corresponding to an eigenvalue form an <u>eigen space.</u>

If dim $=1$, it is non-degenerate.

If dim >1, it is degenerate.

Now,
$$\Omega|v\rangle = \omega|v\rangle$$
$$\Rightarrow (\Omega - \omega I)|v\rangle = |0\rangle$$

Assuming $(\Omega - \omega I)^{-1}$ exists, operating it on both sides,

$$|v\rangle = (\Omega - \omega I)^{-1}|0\rangle, \text{ impossible since } |v\rangle \neq |0\rangle.$$
$$= |0\rangle$$

Thus, $|\Omega - \omega I| = 0$.

$\longrightarrow$ Characteristic eq^n for determining eigenvalues.

Eg. Let $A = \begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix}$. Find its eigenvalues and eigenvectors.

$\rightarrow$ $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -5-\lambda & 2 \\ -7 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (\lambda+5)(\lambda-4) + 14 = 0$$

$$\Rightarrow \lambda^2 + \lambda - 6 = 0$$

$$\Rightarrow \lambda = -3, 2$$

$\therefore$ Eigenvalues $\lambda_1 = -3$, $\lambda_2 = 2$.

$\underline{\lambda_1 = -3}$

$$\begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -5x + 2y \\ -7x + 4y \end{pmatrix} = \begin{pmatrix} -3x \\ -3y \end{pmatrix}$$

$$\begin{aligned} -5x + 2y &= -3x \\ -7x + 4y &= -3y \end{aligned} \quad \Rightarrow x = y$$

$\therefore$ Eigenvectors are $t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, t is an arbitrary scalar.

$\underline{\lambda_2 = 2}$

$$\begin{pmatrix} -5 & 2 \\ -7 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} -5x + 2y \\ -7x + 4y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$\begin{aligned} -5x + 2y &= 2x \\ -7x + 4y &= 2y \end{aligned} \quad \begin{aligned} &\Rightarrow 7x = 2y \\ &\Rightarrow \frac{x}{y} = \frac{2}{7} \end{aligned}$$

$\therefore$ Eigenvectors are $t\begin{pmatrix} 2 \\ 7 \end{pmatrix}$, t is an arbitrary scalar.

$\boxed{\text{Ex.}}$ Consider $\Omega = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$. $(\Omega : \mathbb{R}^3 \to \mathbb{R}^3)$

$\to$ $|\Omega - \lambda I| = 0$

$\Rightarrow (\lambda - 2)^2 \lambda = 0$

$\Rightarrow \lambda = 0, 2, 2$ (Degenerate)

$\underline{\lambda = 0}$

Corresponding eigenvector $= t\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

$\underline{\lambda = 2,}$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$-x + z = 0 \quad \text{— (i)}$$
$$0 = 0 \quad \text{— (ii)}$$
$$x - z = 0 \quad \text{— (iii)}$$

The conditions ~~$x_1 = x_3$ and $x_2 = 0$~~ $x = z$ and $y =$ arbitrary defines an ensemble of vectors orthogonal to $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Two-fold degeneracy leads us to two-dimensional eigenspace.

We can take two orthogonal eigenvectors,

$$t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, t\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

We have produced 3 orthogonal eigenvectors,

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ spanning the domain $\mathbb{R}^3$.

Ex. $\Omega = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$. ($\Omega : \mathbb{R}^2 \to \mathbb{R}^2$)

$\rightarrow$ $|\Omega - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 4-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (\lambda-2)(\lambda-4) + 1 = 0$

$\Rightarrow \lambda^2 - 6\lambda + 9 = 0$

$\Rightarrow (\lambda-3)^2 = 0$

$\Rightarrow \underline{\lambda = 3, 3}$

$$\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3 \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\Rightarrow 4x_1 + x_2 = 3x_1$,

$-x_1 + 2x_2 = 3x_2$

$\Rightarrow \frac{x_1}{x_2} = -1$

$\therefore$ Only one eigenvector of the form $t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. It doesn't span the domain $\mathbb{R}^2$.

$\boxed{\text{Ex:-}}$ If $\Omega = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ (Field is $\mathbb{C}$), what are the eigenvalues and eigenvectors?

$\boxed{\text{Thm:-}}$ An operator is Hermitian iff its eigenvalues are real.

$\boxed{\text{Pf:-}}$ Suppose $\Omega|\omega\rangle = \omega|\omega\rangle \Rightarrow \langle\omega|\Omega|\omega\rangle = \omega\langle\omega|\omega\rangle$

$$\Rightarrow \langle\omega|\Omega^{+} = \langle\omega|\,\omega^{*}$$

$$\Rightarrow \langle\omega|\Omega^{+}|\omega\rangle = \langle\omega|\,\omega^{*}|\omega\rangle$$

$$= \omega^{*}\langle\omega|\omega\rangle$$

Now, $\omega = \omega^{*}$ iff $(\omega-\omega^{*})\langle\omega|\omega\rangle = 0$ [$\because \langle\omega|\omega\rangle > 0 \;\forall\; |\omega\rangle \neq |0\rangle$]

$$\Rightarrow \langle\omega|\Omega-\Omega^{+}|\omega\rangle = 0$$

$$\Rightarrow \Omega = \Omega^{+}, \text{ i.e. } \Omega \text{ is hermitian}$$ [Proved]

$\boxed{\text{Thm:}}$ Eigenvectors of normal operator with distinct eigenvalues are different. (Requires to prove if N is normal and if $N|\lambda\rangle = \lambda|\lambda\rangle$ then $N^\dagger|\lambda\rangle = \lambda^*|\lambda\rangle$)

Diagonalization:-

Spectral theorem: If N is a $n \times n$ normal matrix, i.e. $NN^\dagger = N^\dagger N$ then $\exists$ an eigenbasis in which N is diagonal. $[N: V^n \to V^n]$

Pf: The n^{th} degree polynomial eq^n,

$$|N - \lambda I| = 0$$ must have at least one solution $\in \mathbb{C}$.

Suppose a solution is λ_1. Let corresponding eigenvector be $|\lambda_1\rangle$, i.e. $N|\lambda_1\rangle = \lambda_1|\lambda_1\rangle$.

Consider the vector sub-space $\perp$ to $|\lambda_1\rangle$.

Suppose $|\lambda_1\rangle$ is normalized to unity $[\langle\lambda_1|\lambda_1\rangle = 0]$.

If a basis of $|\lambda_1\rangle^\perp$ is $\{|e_1\rangle, \ldots, |e_{n-1}\rangle\}$, then

$$\langle e_j|N|\lambda_1\rangle = \lambda_1\langle e_j|\lambda_1\rangle$$
$$= 0$$

$$\langle\lambda_1|N|e_j\rangle = \langle N^\dagger\lambda_1|e_j\rangle$$

Since N is normal, $N^\dagger|\lambda_1\rangle = \lambda_1^*|\lambda_1\rangle$

$$\Rightarrow \langle N^\dagger\lambda_1| = \langle\lambda_1|\lambda_1$$

$$\therefore \langle N^\dagger\lambda_1|e_j\rangle = \langle\lambda_1|\lambda_1|e_j\rangle = \lambda_1\langle\lambda_1|e_j\rangle$$
$$= 0.$$

$$\therefore N = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ \vdots & & M & & \\ 0 & & & & \end{pmatrix}, \quad N^\dagger = \begin{pmatrix} \lambda_1^* & 0 & 0 & \cdots & 0 \\ 0 & & & & \\ \vdots & & M^\dagger & & \\ 0 & & & & \end{pmatrix}$$

$$NN^{+} = \begin{pmatrix} |\lambda_1|^2 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & MM^{+} & \\ 0 & & & \end{pmatrix}, \quad N^{+}N = \begin{pmatrix} |\lambda_1|^2 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & M^{+}M & \\ 0 & & & \end{pmatrix}.$$

$NN^{+} = N^{+}N \Rightarrow MM^{+} = M^{+}M$, i.e.

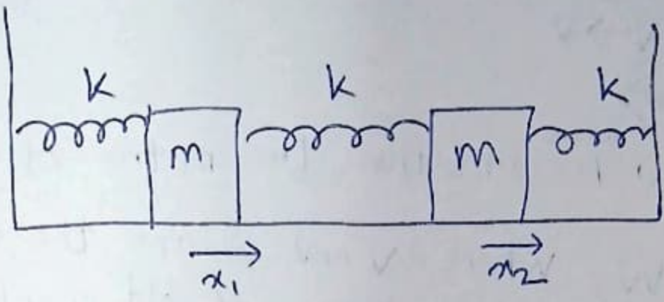
M is also normal.

So, we proceed in this manner and we get an orthonormal basis of eigenvectors of N in which $N = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of N.

[illegible] / Diagonalization

Thm:- If two normal operators are commuting, i.e. $\Lambda\Omega = \Omega\Lambda$, then they are simultaneously diagonalizable.

Solving a physical system with eigenvectors:



Consider the spring-mass system above. Suppose the blocks are displaced by x_1 and x_2 from their equilibrium position respectively.

Using Newton's Laws, we can show (Exercise!) that

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2)$$

$$m\ddot{x}_2 = -kx_2 - k(x_2 - x_1)$$

Thus, $\ddot{x}_1 = -\frac{k}{m}(2x_1 - x_2)$

$\ddot{x}_2 = -\frac{k}{m}(-x_1 + 2x_2)$

$$\therefore \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\frac{k}{m}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{—} \circledast$$

Now, $\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = 1, 3$

$\underline{\lambda = 1}$ $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ [Normalized to unity]

$\underline{\lambda = 3}$

$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 3\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Observe that $\circledast$ has been expressed in the basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Thus, in new basis,

$$\begin{pmatrix} \ddot{x}_1' \\ \ddot{x}_2' \end{pmatrix} = -\frac{k}{m}\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} \quad \text{———} \quad ☆'$$

$$\left.\begin{aligned} \ddot{x}_1' &= -\frac{k}{m} x_1' \\ \ddot{x}_2' &= -\frac{3k}{m} x_2' \end{aligned}\right\} \longrightarrow \text{Our familiar simple SHM equation.}$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = A^{+}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{x_1+x_2}{\sqrt{2}} \\ \frac{x_1-x_2}{\sqrt{2}} \end{pmatrix}.$$

☆' tells us that we can imagine x_1' and x_2' to be two independent harmonic oscillators [Since $\ddot{x}_1' = -\frac{k}{m}x_1'$, $\ddot{x}_2 = -\frac{3k}{m}x_2'$], so eqn. of x_1' and x_2' are completely independent of each other.] So, if the system is left in a pure state x_1' $\left(=\left(\frac{1}{\sqrt{2}}\right)x_1 + x_2\left(\frac{1}{\sqrt{2}}\right)\right)$, that is both move $\rightarrow \rightarrow$ by same amount), it will keep oscillating like this. Similarly, if it left in a pure state x_2' $\left(=\left(\frac{1}{\sqrt{2}}\right)x_1 + \left(-\frac{1}{\sqrt{2}}\right)x_2\right)$ that is they move $\leftarrow \rightarrow$ by same amount, it will also keep oscillating like this.

These are the <u>normal modes</u>.

The corresponding normal frequencies ω_1 and ω_2 are given by

$$\omega_1^2 = +\frac{k}{m} \Rightarrow \omega_1 = \sqrt{\frac{k}{m}}$$

$$\omega_2^2 = \frac{3k}{m} \Rightarrow \omega_2 = \sqrt{\frac{3k}{m}}$$

Thus, we reduced a system of coupled oscillator to a system of simpler, uncoupled oscillators using Linear Algebra!

Infinite Dimensional Vector Space:-

Suppose we have a continuous line x, and at each value of x, value of function $f(x)$ is specified. We can approximate f in a range $[a, b]$ by taking large no. of discrete points $x_1, x_2, \dots, x_n$ and evaluating $f(x)$ at each point.

Then, $$|f\rangle = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}.$$

Basis vectors are $$|x_i\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i^{th} \text{ place.}$$

They satisfy (i) $\langle x_i | x_j \rangle = \delta_{ij}$ (orthogonality)

(ii) $\sum_{i=1}^{n} |x_i\rangle \langle x_i| = I$ (completeness)

Inner product $\langle f|g\rangle = \sum_i f^*(x_i)\, g(x_i)$

When we tend $n \to \infty$ (cont. limit), it approaches infinity. So, we re-define,

$$\langle f|g\rangle = \sum_i f^*(x_i)\, g(x_i)\, \Delta, \quad \Delta = \frac{b-a}{n+1}.$$

$$\therefore \langle f|g\rangle = \int_a^b f^*(x)\, g(x)\, dx \quad \text{as } n \to \infty.$$

$$\langle f|f\rangle = \int_a^b |f(x)|^2\, dx$$

The completeness criteria leads to

$$\int_a^b |x'\rangle\langle x'|\,dx' = 1$$

Now, $|f\rangle = \int_a^b |x'\rangle\langle x'|f\rangle\,dx'$

$$\langle x|f\rangle = \int_a^b \langle x|x'\rangle\langle x'|f'\rangle\,dx'.$$

But, $\langle x|x'\rangle = 0$ when $x \neq x'$.

$$\langle x|f\rangle = \int_{x-\varepsilon}^{x+\varepsilon} \langle x|x'\rangle\langle x'|f'\rangle\,dx'.$$

$\langle x|x'\rangle = \delta(x-x')$ [Dirac delta function]

$\boxed{\text{Properties:}}$

(i) $\displaystyle\int_{-\infty}^{\infty} \delta(x-x')\,f(x')\,dx' = f(x)$

(ii) $\delta(x) = \delta(-x)$

(iii) $\dfrac{d}{dx}\delta(x-x') = -\,\delta(x-x')$